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Elementary Construction of the Quiver of the Mackey Algebra for Groups with Cyclic Normal p-Sylow Subgroup

ALFRED WIEDEMANN*

Mathematisches Institut B der Universität Stuttgart, Stuttgart, Germany

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INTRODUCTION

Through recent investigations of J. Thévenaz and P. J. Webb [4–8], the concept of Mackey functors, being originally introduced by J. A. Green [3] and studied by A. W. M. Dress, T. Yoshida, H. Sasaki, and others, has regained considerable interest. An important reason for this is a new interpretation of a conjecture of Alperin in terms of Mackey functors in [8] using results of [7]. Given a commutative ring k, by its definition, it is immediate that the category of Mackey functors over k for a finite group G is an abelian category, even that it is the category of k-representations of a quiver with relations [8]. Namely a Mackey functor for G associates to each subgroup of G a k-module and moreover certain k-linear maps between these spaces satisfying certain relations involving G. Thus a Mackey functor for G can be interpreted as a module over a path algebra of a quiver with relations, the so-called Mackey algebra [6, 9]. The main difference to the usual representation theory of quivers, and thus the main problem of interpreting Mackey functors as such representations, lies in the fact that the maps between these spaces do not lie in the radical of this path algebra, and this path algebra is not basic in general. So it is a natural problem to compute the quiver in the usual sense for the category of Mackey functors for a group G in particular in the case where k is a field of characteristic p dividing the order of G.

The question of when there are up to isomorphism only finitely many indecomposable Mackey functors for G led to recent investigations of Thévenaz and Webb solving this problem in general, and it also led to the investigations we present in this paper which presents in a special situation, namely if G has a normal cyclic p-Sylow subgroup, an explicit computation of the quiver of the now so-called Mackey algebra. By definition, the Mackey algebra is the basic k-algebra whose category of finitely generated

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modules is equivalent to the category of Mackey functors for G with the property that all k-spaces involved are finite dimensional over k.

The word "elementary" in the title is meant to emphasize the fact that we do not use the most recent results of Thévenaz and Webb on the block decomposition of the Mackey algebra [9]. We only use basic properties of Mackey functors coming from their definition, special properties of the groups with cyclic normal p-Sylow subgroup, the results of [6, 7], and easy quiver theoretic arguments. Because of the elementary and explicit nature of our argument, we hope that besides the forthcoming more general results of Thévenaz and Webb, the computation of the quiver of the Mackey algebra we present here is of independent interest.

To state our result let k be an algebraically closed field of characteristic $p>0$, let G be a finite group with a cyclic normal p-Sylow subgroup $P=C_{p^n}$ generated by an element $a\in G$ of order p^n, let H be a complement of G, that is, H is a maximal p'-subgroup of G, and $G=P\rtimes H$ is the semi-direct product of P with H, and let $C=C_H(P)$ be the centralizer of P in H. From [7] we recall that the simple Mackey functors are in one-to-one correspondence to the conjugacy classes of pairs (G', E), G' a subgroup of G and E a simple module of the group algebra $k(N_G(G')/G')$, $N_G(G')$ being the normalizer of G' in G. Then the result is the following.

THEOREM. *The simple Mackey functors associated with those subgroups of G which are not contained in $Z=P\times C$ are projective and injective. The simple Mackey functors associated with the subgroups of Z form in the quiver of the Mackey algebra connected components of the form $K_{n,e}$, where e divides $p-1$, and $K_{n,e}$ is the quiver with $e(n+1)$ vertices and relations constructed as follows. The vertices are labelled by pairs of integers (i,j), $0\leqslant i\leqslant e$ and $0\leqslant j\leqslant n$, where the vertices $(0,j)$ and (e,j) have to be identified. There are families of arrows r, t, and α as indicated in the diagram to which we attach for simplicity only one index*:

$$K_{n,e}:\quad\begin{tikzcd}
(0,n) \arrow[d, shift right, "r_n"'] & (1,n) \arrow[d, shift right, "r_n"'] & (2,n) \arrow[d, shift right, "r_n"'] & & (e-1,n) \arrow[d, shift right, "r_n"'] & (e,n) \arrow[d, shift right, "r_n"'] \\
(0,n-1) \arrow[u, shift right, "t_n"'] \arrow[r, "\alpha_{n-1}"'] & (1,n-1) \arrow[u, shift right, "t_n"'] \arrow[r, "\alpha_{n-1}"'] & (2,n-1) \arrow[u, shift right, "t_n"'] \arrow[r, "\alpha_{n-1}"'] & \cdots \arrow[r, "\alpha_{n-1}"'] & (e-1,n-1) \arrow[u, shift right, "t_n"'] \arrow[r, "\alpha_{n-1}"'] & (e,n-1) \arrow[u, shift right, "t_n"'] \\
\vdots & \vdots & \vdots & & \vdots & \vdots \\
(0,2) \arrow[d, shift right, "r_2"'] \arrow[r, "\alpha_2"'] & (1,2) \arrow[d, shift right, "r_2"'] \arrow[r, "\alpha_2"'] & (2,2) \arrow[d, shift right, "r_2"'] \arrow[r, "\alpha_2"'] & \cdots \arrow[r, "\alpha_2"'] & (e-1,2) \arrow[d, shift right, "r_2"'] \arrow[r, "\alpha_2"'] & (e,2) \arrow[d, shift right, "r_2"'] \\
(0,1) \arrow[u, shift right, "t_2"'] \arrow[d, shift right, "r_1"'] \arrow[r, "\alpha_1"'] & (1,1) \arrow[u, shift right, "t_2"'] \arrow[d, shift right, "r_1"'] \arrow[r, "\alpha_1"'] & (2,1) \arrow[u, shift right, "t_2"'] \arrow[d, shift right, "r_1"'] \arrow[r, "\alpha_1"'] & \cdots \arrow[r, "\alpha_1"'] & (e-1,1) \arrow[u, shift right, "t_2"'] \arrow[d, shift right, "r_1"'] \arrow[r, "\alpha_1"'] & (e,1) \arrow[u, shift right, "t_2"'] \arrow[d, shift right, "r_1"'] \\
(0,0) \arrow[u, shift right, "t_1"'] \arrow[r, "\alpha_0"'] & (1,0) \arrow[u, shift right, "t_1"'] \arrow[r, "\alpha_0"'] & (2,0) \arrow[u, shift right, "t_1"'] \arrow[r, "\alpha_0"'] & \cdots \arrow[r, "\alpha_0"'] & (e-1,0) \arrow[u, shift right, "t_1"'] \arrow[r, "\alpha_0"'] & (e,0) \arrow[u, shift right, "t_1"']
\end{tikzcd}$$

Additionally, all possible relations of the following form are satisfied:

$$\alpha_{i-1}r_i = r_i\alpha_i, \qquad \alpha_i t_i = t_i\alpha_{i-1} \qquad \textit{for} \quad i = 1, ..., n-1,$$

$$\alpha_{n-1}r_n = t_n\alpha_{n-1} = 0,$$

and

$$r_i t_i = \alpha_{i-1}^{(p-1)p^{n-i}} \qquad \textit{for} \quad i = 1, ..., n.$$

For more detailed information on the number of such components and the meaning of the parameter e, we refer to Proposition 3. Note also that for $p = 2$, the arrow α_{n-1} disappears in accordance with the special role played by the prime 2 in Lemma 3 of Section 2.

The quiver of the Mackey algebra was well known before by Thévenaz and Webb in special situations, in particular for G the cyclic group of order p. I am grateful to J. Thévenaz for showing me this result and for providing me with further informations on their joint work. I also thank Wolfgang Kimmerle for helpful conversations during the preparation of this paper.

The proof of the Theorem is spread out over Sections 2 and 3.

We want to conclude this Introduction with some remarks on the representation type of the Mackey algebra.

(1) The quiver $K_{n,e}$ is an e-fold covering of the quiver $K_{n,1}$ in the sense of the covering theory by Bongartz and Gabriel [1]. Therefore, the Mackey algebra of a group G as in the Theorem is of finite representation type if and only if

$$K_{n,1} = \begin{tikzcd} (0,n) \arrow[d, shift right, "r_n"'] \\ (0,n-1) \arrow[u, shift right, "t_n"'] \arrow[loop right, "\alpha_{n-1}"] \\ \vdots \\ (0,2) \arrow[loop right, "\alpha_2"] \arrow[d, shift right, "r_2"'] \\ (0,1) \arrow[u, shift right, "t_2"'] \arrow[loop right, "\alpha_1"] \arrow[d, shift right, "r_1"'] \\ (0,0) \arrow[u, shift right, "t_1"'] \arrow[loop right, "\alpha_0"] \end{tikzcd}, \qquad \text{with relations as above,}$$

being the quiver of the Mackey algebra of C_{p^n}, is of finite representation type as a quiver over k. With the results of Fischbacher [2], one easily verifies that this is the case if and only if n is zero or one.

(2) Concerning the representation type of the Mackey algebra in general, the reader is referred to the forthcoming results of Thévenaz and Webb [9]. Let us just recall that for an arbitrary finite group G there is a full embedding of the category of finitely generated kG-modules into the category of Mackey functors for G. Therefore, if the p-Sylow subgroup of G is not cyclic, the Mackey algebra automatically is of infinite representation type. This is the reason why we are only interested here in the case of a cyclic p-Sylow subgroup.

1. Preliminaries and Known Results

Once and for all we fix a field k of characteristic p which we assume for simplicity to be algebraically closed. For a finite group G, a Mackey functor $\mathcal{M}$ for G over k, associates to each subgroup U of G a finite dimensional k-vectorspace $\mathcal{M}(U)$ and for all pairs of subgroups V, U of G with $V \leqslant U$, k-linear maps $r^U_V: \mathcal{M}(U) \to \mathcal{M}(V)$ and $t^U_V: \mathcal{M}(V) \to \mathcal{M}(U)$ called restriction maps, respectively transfer maps, and for all $g \in G$, k-linear maps $c_g(U): \mathcal{M}(U) \to \mathcal{M}({}^gU)$, where ${}^gU = gUg^{-1}$, which are called conjugation maps. Altogether the following axioms must be satisfied [3, 4].

(M1) $r^U_U = t^U_U = c_g(U)$ is the identity for all subgroups U of G and whenever $g \in U$. Moreover, $r^V_W r^U_V = r^U_W$ and $t^U_V t^V_W = t^U_W$ for subgroups $W \leqslant V \leqslant U$ of G.

(M2) $r^{{}^gU}_{{}^gV} c_g(U) = c_g(V)\, r^U_V$ and $t^{{}^gU}_{{}^gV} c_g(V) = c_g(U)\, t^U_V$ whenever $V \leqslant U$ are subgroups of G, and for all $g \in G$.

(M3) $c_{gh}(U) = c_g({}^hU)\, c_g(U)$ for all $g, h \in G$ and subgroups U of G.

(M4) The Mackey axiom: For each subgroup U of G and subgroups V, W of U,

$$r^U_W t^U_V = \sum_{g \in [W \backslash U / V]} t^W_{W \cap {}^gV} (r^{{}^gV}_{W \cap {}^gV} c_g(V)),$$

g running through a system of double coset representatives of U with respect to W and V.

Since there are no confusions possible, we write from now on simply g instead of $c_g(U)$.

It is immediately clear that via the conjugation maps, $\mathcal{M}(U)$ is a $k(N_G(U)/U)$-module.

One obviously has straightforward concepts of Mackey subfunctors, of simple Mackey functors, and of indecomposability of Mackey functors.

Let $\mathcal{M}$ be a Mackey functor for G, and let $\mathfrak{X}$ be a subset of subgroups of G which is closed under taking subgroups and conjugates, for short, $\mathfrak{X}$ is called subconjugately closed. Then one has by [7] the following important Mackey subfunctors for G, namely $\operatorname{Ker} r_{\mathscr{X}}(\mathcal{M})$ and $\operatorname{Im} t_{\mathscr{X}}(\mathcal{M})$ defined by

$$\operatorname{Ker} r_{\mathscr{X}}(\mathcal{M})(U) = \bigcap_{\substack{X \in \mathscr{X} \\ X \leqslant U}} \operatorname{Ker} r_X^U \qquad \text{and} \qquad \operatorname{Im} t_{\mathscr{X}}(\mathcal{M})(U) = \sum_{\substack{X \in \mathscr{X} \\ X \leqslant U}} \operatorname{Im} t_X^U.$$

The simple Mackey functors are completely described and constructed by Thévenaz and Webb [7]. From there, recall that for each simple Mackey functor $\mathscr{S}$ the minimal subgroups on which $\mathscr{S}$ does not vanish form a unique conjugacy class $\{{}^gU \mid g \in G \text{ and } U \text{ a subgroup of } G\}$, and moreover, $\mathscr{S}(U)$ is a simple $k(N_G(U)/U)$-module. In [7], $\mathscr{S}$ is then denoted as $S_{U,E}$ for $E = \mathscr{S}(U)$. The construction of $S_{U,E}$ is completely given in [6, 7]. Especially it is shown that two simple Mackey functors $S_{U,E}$ and $S_{U',E'}$ are isomorphic if and only if both U and U' and E and E' are conjugate under the same element of G. Then U is called a minimal subgroup associated to $\mathscr{S}$.

One of the main results of [7] is the following fact. If G is a p'-group, that is, if p does not divide the order of G, then each Mackey functor $\mathcal{M}$ for G decomposes completely into a direct sum of simple Mackey functors. Moreover, in this situation, for each subconjugately closed subset $\mathscr{X}$ of subgroups of G, one has $\mathcal{M} = \operatorname{Ker} r_{\mathscr{X}}(\mathcal{M}) \oplus \operatorname{Im} t_{\mathscr{X}}(\mathcal{M})$.

2. The Semisimple Part of the Mackey Algebra

In this section we split off the semisimple part of the Mackey algebra of a group with cyclic normal p-Sylow subgroup. The first step described in Lemma 1 and Proposition 1 can be formulated in a slightly more general context.

Lemma 1. *Let N be a normal subgroup of an arbitrary finite group G such that G/N is a p'-group and such that for all subgroups U of G, $U \cap N$ is normal in U. If $\mathscr{X}$ is the set of subgroups of N, then each Mackey functor $\mathcal{M}$ for G decomposes as $\mathcal{M} = \operatorname{Ker} r_{\mathscr{X}}(\mathcal{M}) \oplus \operatorname{Im} t_{\mathscr{X}}(\mathcal{M})$.*

Proof. Clearly, $\mathscr{X}$ is subconjugately closed, and $\operatorname{Ker} r_{\mathscr{X}}(\mathcal{M})$ and $\operatorname{Im} t_{\mathscr{X}}(\mathcal{M})$ are indeed subfunctors of $\mathcal{M}$. Moreover, $\operatorname{Ker} r_{\mathscr{X}}(\mathcal{M})(U) = \operatorname{Ker} r_{U \cap N}^U$ and $\operatorname{Im} t_{\mathscr{X}}(\mathcal{M})(U) = \operatorname{Im} t_{U \cap N}^U$ for each subgroup U of G. For each $x \in \mathcal{M}(U)$, $x - t_{U \cap N}^U(r_{U \cap N}^U(x/|U : U \cap N|))$ lies in $\operatorname{Ker} r_{U \cap N}^U$, namely

$$\begin{aligned}
& r_{U \cap N}^U(x - t_{U \cap N}^U(r_{U \cap N}^U(x/|U : U \cap N|))) \\
&\quad = r_{U \cap N}^U x - \sum_{g \in [U/U \cap N]} g(r_{U \cap N}^U(x/|U : U \cap N|)) \\
&\quad = r_{U \cap N}^U x - r_{U \cap N}^U\Bigg(\sum_{g \in [U/U \cap N]} g(x/|U : U \cap N|)\Bigg) = 0,
\end{aligned}$$

using the axioms (M1)-(M4). Moreover, if $x = t^U_{U\cap N}y$ lies in $\operatorname{Ker} r^U_{U\cap N}(U) \cap \operatorname{Im} t^U_{U\cap N}(U)$, then

$$0 = t^U_{U\cap N}(r^U_{U\cap N}(t^U_{U\cap N}(y))) = \sum_{g \in [U/U\cap N]} t^U_{U\cap N}(g(y))$$
$$= \sum_{g \in [U/U\cap N]} g t^U_{U\cap N} y = |U : U \cap N|\, x,$$

forcing $x = 0$. ∎

PROPOSITION 1. *Let G be a finite group with a normal p-Sylow subgroup P, and let $\mathscr{X}$ be the set of subgroups of G being contained in $Z = C_G(P)$. Then each Mackey functor $\mathscr{M}$ for G decomposes as*

$$\mathscr{M} = \operatorname{Ker} r_{\mathscr{X}}(\mathscr{M}) \oplus \operatorname{Im} t_{\mathscr{X}}(\mathscr{M}).$$

Proof. With $N = Z$, Lemma 1 can be applied. ∎

We now fix the notation for the rest of this paper and recall some well known facts about groups with a normal cyclic p-Sylow subgroup.

Let G be a finite group with normal cyclic p-Sylow subgroup $P = C_{p^n}$ being generated by an element $a \in G$ of order p^n. Let H, $C = C_H(P)$, and $Z = P \times C = C_G(P)$ be as in the Introduction. Then H/C is a cyclic group, say of order e, with e dividing $p-1$. Moreover, C is normal in G, and G/C is a Frobenius group with kernel P and complement H/C. We furthermore need the following properties of G which can be verified easily.

For each nontrivial element $b \in P$ and each p'-subgroup U of G,

$$U \cap {}^bU = U \cap C,$$

and more generally, denoting by C_{p^i} the subgroup of G of order p^i for $i = 0, 1, ..., n$, one has

$$(C_{p^i} \rtimes U) \cap {}^b(C_{p^i} \rtimes U) = C_{p^i} \times (C \cap U).$$

Furthermore, $N_G(C') = N_G(C_{p^i} \times C') = C_{p^n} \rtimes N_H(C')$ for all i and arbitrary subgroups C' of C.

PROPOSITION 2. *Let $\mathscr{M}$ be a Mackey functor for G such that $\mathscr{M}(V) = 0$ for all subgroups V of $Z = C_G(P)$. Then $\mathscr{M}$ is completely decomposable into simple Mackey functors.*

Proof. Note that with the notation of Proposition 1, $\mathscr{M} = \operatorname{Ker} r_{\mathscr{X}}(\mathscr{M})$ holds. By [6, 7] as we mentioned in the Introduction, if G is a p'-group, $\mathscr{M}$ decomposes completely. By induction on n, we may then suppose that this result is true for the group $\bar{G}=G/C_p$. Then again by the above result of Thévenaz and Webb, the full subquiver of the quiver of the Mackey algebra formed by the simple Mackey functors whose associated minimal subgroups contain C_p but are not contained in Z is discrete. Since $a^{p^{n-1}}$ acts trivially on the spaces $\mathcal{M}(U)$ for a subgroup U containing C_p, note for this remark that the relations coming from the axioms (M1)–(M4) involving the subgroups of this form are exactly the same as those for a Mackey functor for $\bar{G}$ for the corresponding subgroups of $\bar{G}$. Putting $\mathscr{Y}$ as the set of p'-subgroups not contained in C, and supposing that $\mathcal{M}=\operatorname{Ker} r_{\mathscr{X}}(\mathcal{M})$, $\mathcal{M}$ decomposes as $\mathcal{M}=\operatorname{Ker} r_{\mathscr{Y}}(\mathcal{M})\oplus \operatorname{Im} t_{\mathscr{Y}}(\mathcal{M})$. Namely let U be any subgroup of G. Then $U=C_{p^i}\rtimes W$ for a p'-subgroup W. Only the case where W is not contained in C is of interest. Then for each $x\in\mathcal{M}(U)$, $x-t^U_W r^U_W x$ lies in $\operatorname{Ker} r_{\mathscr{Y}}(\mathcal{M})$ because all p'-subgroups of U are contained in a conjugate of W of the form ${}^bW$ for some $b\in C_{p^i}$, and

$$\begin{aligned} r^U_{{}^bW}(x-t^U_W r^U_W x) &= r^U_{{}^bW}x - r^U_{{}^bW}(t^U_W(r^U_W x)) \\ &= r^U_{{}^bW}x - \sum_{g\in[{}^bW\backslash U/W]} t^{{}^bW}_{{}^gW\cap{}^bW}(r^{{}^gW}_{{}^gW\cap{}^bW}(g(r^U_W x))) \\ &= r^U_{{}^bW}x - b(r^U_W x) = r^U_{{}^bW}x - r^U_{{}^bW}(bx)=0. \end{aligned}$$

For this note that since $\mathcal{M}(V)=0$ for all subgroups V of C, and since ${}^bW\cap{}^gW$ is contained in C if ${}^bW\neq{}^gW$, exactly one term of the sum above does not vanish. Moreover, one uses that $b\in U$.

Now let $x\in\operatorname{Ker} r_{\mathscr{Y}}(\mathcal{M})(U)\cap\operatorname{Im} t_{\mathscr{Y}}(\mathcal{M})(U)$. Again we suppose that $U=C_{p^i}\rtimes W$ for a p'-subgroup W being not contained in C. Since the maximal subgroups U which lie in $\mathscr{Y}$ are all conjugate under C_{p^i}, and by (M2), we may assume that $x=t^U_W y$ for some $y\in\mathcal{M}(W)$. Then it follows directly by the Mackey axiom and noting again that ${}^bW\cap W\leqslant C$ for each nontrivial $b\in C_{p^n}$ that $0=r^U_W x=r^U_W t^U_W y=y$, forcing $x=0$.

Now assume that $\mathcal{M}$ is an indecomposable Mackey functor for G. If $\mathcal{M}=\operatorname{Ker} r_{\mathscr{Y}}(\mathcal{M})$, then by induction, $\mathcal{M}$ is a simple Mackey functor for $\bar{G}$, and therefore, it is also simple as a Mackey functor for G.

Finally, it remains the case where $\mathcal{M}=\operatorname{Im} t_{\mathscr{Y}}(\mathcal{M})$. For each group $W\in\mathscr{Y}$, it is immediately clear that $t^{C_{p^n}\rtimes W}_W$ is an isomorphism between $\mathcal{M}(W)$ and $\mathcal{M}(C_{p^n}\rtimes W)$ with inverse $r^{C_{p^n}\rtimes W}_W$. Any nontrivial decomposition of the restriction of $\mathcal{M}$ to the subset of subgroups of G containing C_{p^n}, would therefore provide a nontrivial decomposition of $\mathcal{M}$ itself. To see this, note that all conjugates of W are of the form ${}^bW$ for $b\in C_{p^n}$, and the conjugation map $c_b(W)\colon\mathcal{M}(W)\to\mathcal{M}({}^bW)$ coincides with the composition $r^{C_{p^n}\rtimes W}_{{}^bW}t^{C_{p^n}\rtimes W}_W$. This shows in particular, that any decomposition as mentioned above would be respected by the conjugation action of C_{p^n} on $\mathcal{M}$. Now with the previous arguments, the restriction of $\mathcal{M}$ to the subset of groups containing C_{p^n} can be viewed as a simple Mackey functor for H. The description of the simple Mackey functors by Thévenaz and Webb [7] implies that $\mathcal{M}$ itself is a simple Mackey functor for G. ∎

3. The Nontrivial Components of the Mackey Quiver

We now consider the simple Mackey functors for G which are generated via transfer maps from their restriction to the subgroups of Z. They are of the form $\mathcal{S} = \operatorname{Im} t_{\mathcal{X}}(\mathcal{S})$ in the notation of Proposition 1. We keep the notation used so far.

By the support of a Mackey functor $\mathcal{M}$ for G, we clearly mean the set of subgroups U of G with $\mathcal{M}(U) \neq 0$. We denote by $\operatorname{Mack}(G)$ the abelian category of Mackey functors for G.

Lemma 2. *Let $\mathcal{S}'$ and $\mathcal{S}''$ be simple Mackey functors for G whose associated minimal subgroups are contained in C. Then they have a nontrivial extension group $\operatorname{Ext}^1_{\operatorname{Mack}(G)}(\mathcal{S}'', \mathcal{S}')$ only if they have the same minimal subgroups.*

Proof. We consider an extension $0 \to \mathcal{S}' \to \mathcal{T} \to \mathcal{S}'' \to 0$ of Mackey functors for G and denote by C', respectively C'', a minimal subgroup associated to $\mathcal{S}'$ and to $\mathcal{S}''$, respectively. Since all these three functors are induced via transfer maps from subgroups of Z, this sequence splits if and only if the restriction of this sequence to $\mathcal{X}$ splits. We denote by $\tilde{\mathcal{S}}'$, $\tilde{\mathcal{S}}''$, $\tilde{\mathcal{T}}$ the corresponding restrictions of $\mathcal{S}'$, $\mathcal{S}''$, $\mathcal{T}$ to $\mathcal{X}$. Recall that a denotes a generator for C_{p^n}. As an element of p-power order, a acts trivially on all simple modules over $N_G(C')/C'$ and $N_G(C'')/C''$, implying that $(a-1)\tilde{\mathcal{S}}' = (a-1)\tilde{\mathcal{S}}'' = 0$. Note that conjugation with a leaves fixed all the subgroups in $\mathcal{X}$. Moreover, the construction of the simple Mackey functors in [7] shows immediately that $\tilde{\mathcal{S}}'$, $\tilde{\mathcal{S}}''$, and $\tilde{\mathcal{T}}$ do not vanish only on subgroups of C.

If $(a-1)\tilde{\mathcal{T}} = 0$, then a also acts trivially on $\mathcal{T}$, which then decomposes as the restriction to $\mathcal{X}$ of a Mackey functor for H, and moreover, any H-splitting of the above sequence also is a G-splitting, since G acts via the canonical map onto G/C_{p^n}. Hence, $\tilde{\mathcal{T}}$ decomposes, and therefore also $\mathcal{T}$ decomposes. In the other case, since C_{p^n} is normal, $(a-1)\tilde{\mathcal{T}}$ is strictly contained in $\tilde{\mathcal{T}}$, which must in the case where $\tilde{\mathcal{S}}'$ and $\tilde{\mathcal{S}}''$ have different associated minimal subgroups, and therefore are not isomorphic, coincide with $\tilde{\mathcal{S}}'$, that is, $(a-1)\tilde{\mathcal{T}} = \tilde{\mathcal{S}}'$. This implies that $\mathcal{S}''(C')$ does not vanish, and we may assume that C'' is strictly contained in C'. Now let $\mathcal{C}$ be the subconjugacy closure of the set of proper subgroups of C'. Since H is a p'-group, and since $\tilde{\mathcal{T}}$ can be viewed as the restriction of a Mackey functor for H, because its support is contained in the set of subgroups of C, by [7], $\tilde{\mathscr{T}}$ decomposes as $\tilde{\mathscr{T}} = \widetilde{\operatorname{Ker} r_{\mathscr{C}}(\mathscr{T})} \oplus \widetilde{\operatorname{Im} t_{\mathscr{C}}(\mathscr{T})}$. This also is a decomposition over G since $\mathscr{C}$ is subconjugately closed in G. Furthermore, it is clear that $\widetilde{\operatorname{Im} t_{\mathscr{C}}(\mathscr{T})}$ does not vanish, and $\tilde{\mathscr{S}}'$ is contained in $\widetilde{\operatorname{Ker} t_{\mathscr{C}}(\mathscr{T})}$ because the support of $\mathscr{S}'$ does not intersect $\mathscr{C}$. Therefore, $\tilde{\mathscr{T}}$ decomposes as the restriction of a Mackey functor for G on $\mathscr{X}$, and therefore, the originally given exact sequence of Mackey functors also splits, as was to be shown. ∎

LEMMA 3. *Let C' be a subgroup of C, let V and W be simple $k(N_G(C')/C')$-modules, and let $\mathscr{S}'$ respectively $\mathscr{S}''$ be simple Mackey functors with associated minimal subgroup C' satisfying $\mathscr{S}'(C') = V$ and $\mathscr{S}''(C') = W$. Then the evaluation at C' provides an isomorphism*

$$\operatorname{Ext}^1_{\mathrm{Mack}(G)}(\mathscr{S}'', \mathscr{S}') \to \operatorname{Ext}^1_{k(N_G(C')/C')}(W, V)$$

whenever $p \geqslant 3$ or $n \geqslant 2$. For $p = 2$ and $n = 1$, one always has $\mathscr{S}'' \cong \mathscr{S}'$, $V \cong W$, and $\operatorname{Ext}^1_{\mathrm{Mack}(G)}(\mathscr{S}', \mathscr{S}') = 0$, whereas $\operatorname{Ext}^1_{k(N_G(C')/C')}(V, V) \cong k$.

Proof. Let $0 \to \mathscr{S}' \to \mathscr{T} \to \mathscr{S}'' \to 0$ be an exact sequence with the given $\mathscr{S}'$ and $\mathscr{S}''$. Then $\mathscr{T}$ has subfunctors $\operatorname{Im} t_{\mathscr{C}}(\mathscr{T})$ and $\operatorname{Ker} r_{\mathscr{C}}(\mathscr{T})$ for $\mathscr{C}$ being the subconjugacy closure of all subgroups of C'. Clearly, $\operatorname{Im} t_{\mathscr{C}}(\mathscr{T})(C') = \mathscr{T}(C')$ and $\operatorname{Ker} r_{\mathscr{C}}(\mathscr{T})(C') = 0$. Since $\mathscr{T}$ has the two composition factors $\mathscr{S}'$ and $\mathscr{S}''$ with associated minimal subgroup C', this forces $\operatorname{Im} t_{\mathscr{C}}(\mathscr{T}) = \mathscr{T}$ and $\operatorname{Ker} r_{\mathscr{C}}(\mathscr{T}) = 0$. This implies that $\mathscr{T}(C')$ decomposes nontrivially if and only if $\mathscr{T}$ does so, equivalently, the sequence $0 \to \mathscr{S}' \to \mathscr{T} \to \mathscr{S}'' \to 0$ splits. Therefore, the evaluation at C' induces a monomorphism between the extension groups.

Let conversely $0 \to V \to T \to W \to 0$ be a nontrivial extension of the given V and W and a $k(N_G(C')/C')$-module T. It is well known that then $\operatorname{Ext}^1_{k(N_G(C')/C')}(W, V) \cong k$. We therefore must construct an indecomposable Mackey functor $\mathscr{T}$ with subfunctor $\mathscr{S}'$ and $\mathscr{T}/\mathscr{S}' \cong \mathscr{S}''$. For the technical details we now refer to [6, 7]. Following the construction of the simple Mackey functors there, let first $\operatorname{Im} t_{\mathscr{C}}(FP_T)$ be the smallest subfunctor of the fixed point functor FP_T of T as a Mackey functor for $N_G(C')/C'$ with evaluation T at its unit element C'/C'. Recall that for a subgroup J of $N_G(C')/C'$ the set $FP_T(J)$ is given by the set T^J of J-fixed points of T. Furthermore let $(\operatorname{Inf}^{N_G(C')}_{N_G(C')/C'} \operatorname{Im} t_{\mathscr{C}}(FP_T)) \uparrow^G_{N_G(C')}$ be the induction from $N_G(C')$ to G of the inflation of $\operatorname{Im} t_{\mathscr{C}}(FP_T)$ to $N_G(C')$. By Proposition 1, this functor has an indecomposable direct summand $\mathscr{T}$ being generated via transfer maps coming from subgroups of Z and satisfying $\mathscr{T}(C') = T$. Moreover, if $p \geqslant 3$ or $n \geqslant 2$, T is annihilated by $\sum_{i=0}^{p-1} a^{ip^{n-1}} = (a^{p^{n-1}} - 1)^{p-1} = (a-1)^{(p-1)p^{n-1}}$. Since for FP_T the transfer maps are all given by relative traces, all transfer maps ending in subgroups with a nontrivial p-Sylow subgroup vanish. The support of $\mathcal{T}$ therefore only consists of p'-subgroups. By Lemma 1, $\mathcal{T}$ only has simple composition factors with associated minimal subgroup C', and since $\mathcal{T}(C') = T$, $\mathcal{T}$ has exactly $\mathcal{S}'$ and $\mathcal{S}''$ as composition factors. By construction, clearly $\mathcal{S}'$ is a subfunctor of $\mathcal{T}$, and hence $\mathcal{T}/\mathcal{S}' \cong \mathcal{S}''$ as desired. For $p=2$, $n=1$, and a the generator of $C_2 \leqslant G$, the action of $a-1$ on $\mathcal{S}'(C')$ coincides by the Mackey axiom with the composition $r_{C'}^{C_2\times C'} t_{C'}^{C_2\times C'}$. This provides the last statement. ∎

Now let $\mathcal{S}'$ and $\mathcal{S}''$ be simple Mackey functors for G, and suppose that $\mathcal{S}'$ has a subgroup C' of C as an associated minimal subgroup and that $\mathcal{S}''$ has $C_{p^i} \times C''$ as an associated minimal subgroup for some $i \geqslant 1$ and a subgroup C'' of C. To study extensions of $\mathcal{S}'$ and $\mathcal{S}''$ with each other, it is again, as in the proof of Lemma 2, sufficient to consider their restrictions $\tilde{\mathcal{S}}'$ and $\tilde{\mathcal{S}}''$ to $\mathcal{X}$. Again the support of $\tilde{\mathcal{S}}'$ is contained in the set of subgroups of C, and the support of $\tilde{\mathcal{S}}''$ only contains subgroups of Z having C_{p^i} as the p-Sylow subgroup. Moreover, since C_{p^n} acts trivially on $\tilde{\mathcal{S}}'$ and $\tilde{\mathcal{S}}''$, we can in an obvious way, as we did in the proof of Lemma 2, view both of them as restrictions of Mackey functors for H.

LEMMA 4. *With this notation, the simple Mackey functors $\mathcal{S}'$ and $\mathcal{S}''$ have a nontrivial extension group $\mathrm{Ext}^1_{\mathrm{Mack}(G)}(\mathcal{S}'', \mathcal{S}')$ if and only if $i=1$, and $\tilde{\mathcal{S}}'$ and $\tilde{\mathcal{S}}''$ are isomorphic when they are viewed as restrictions of Mackey functors for H. In this case, both $\mathrm{Ext}^1_{\mathrm{Mack}(G)}(\mathcal{S}'', \mathcal{S}')$ and $\mathrm{Ext}^1_{\mathrm{Mack}(G)}(\mathcal{S}', \mathcal{S}'')$ are one-dimensional over k.*

Proof. Let $0 \to \mathcal{S}' \to \mathcal{T} \to \mathcal{S}'' \to 0$ be an exact sequence of Mackey functors for G. If $i>1$, then $\mathcal{T}$ vanishes on subgroups having C_p as the p-Sylow subgroup. Therefore, all restriction and transfer maps of $\mathcal{T}$ connecting the spaces attached to a p'-subgroup and to a subgroup of G with C_{p^i} as the p-Sylow subgroup vanish. Hence, $\mathcal{T}$ decomposes into the direct sum of $\mathcal{S}'$ and $\mathcal{S}''$. For $i=1$ and the restrictions $\tilde{\mathcal{S}}'$ and $\tilde{\mathcal{S}}''$ viewed over H as described, both the restriction and transfer maps given by the Mackey functor $\mathcal{T}$ between the spaces attached to the groups J and $C_p \rtimes J$ for $J \leqslant H$ can be interpreted as a morphism of Mackey functors for H. To see this, note that these restriction maps commute clearly with the other restriction maps and with the conjugation maps by (M1) and (M2). Furthermore, the Mackey axiom implies the commutation with the transfer maps. Similarly one argues for the transfer maps. Since $\tilde{\mathcal{S}}'$ and $\tilde{\mathcal{S}}''$ generate also simple Mackey functors for H, we can conclude that they either must be isomorphic over H, or all these restriction and transfer maps must vanish. In the last case, the given exact sequence splits.

Finally, if $i=1$ and $\tilde{\mathcal{S}}'$ and $\tilde{\mathcal{S}}''$ are isomorphic as restrictions of Mackey functors for H, we can choose suitable isomorphisms as restriction maps and zero maps as transfer maps or the other way round. In each case, we get an indecomposable Mackey functor for G with $\mathscr{S}'$ and $\mathscr{S}''$ as composition factors. Moreover, each nontrivial extension of $\mathscr{S}'$ and $\mathscr{S}''$ is determined by an isomorphism between the spaces $\mathscr{S}'(C')$ and $\mathscr{S}''(C_p \times C')$ as simple $k(N_H(C')/C')$-modules representing either the restriction or the transfer map for $\mathscr{T}$. This provides the last statement. ▮

For short, the ordinary quiver of the Mackey algebra of G over k will be called the Mackey quiver of G. In Proposition 3 we shall describe the nontrivial connected components of the Mackey quiver of G. We already know by the results of Section 2 that it contains only simple Mackey functors whose associated minimal subgroups are contained in $Z = C_G(C_{p^n})$.

This finally also will finish the proof of the Theorem stated in the Introduction.

PROPOSITION 3. *The simple Mackey functors for G whose associated minimal subgroups are contained in Z form connected components of the Mackey quiver of the form $K_{n,e}$ as introduced in the Introduction with e dividing $p-1$. These connected components are in a one-to-one correspondence to the set of blocks of the group algebras $k(N_G(C')/C')$, C' running through a system of representatives of the set of conjugacy classes of subgroups C' of C. Moreover, if such a block has e nonisomorphic simple modules, the corresponding component is of type $K_{n,e}$.*

Proof. By induction, one has this result for $\bar{G} = G/C_p$ accordingly, providing a bunch of full subquivers of the Mackey quiver of type $K_{n-1,e}$. For a subgroup C' of C and by Lemma 3, the full subquiver of the Mackey quiver made up from the simple Mackey functors having C' as a minimal subgroup is just the ordinary quiver of the group algebra $k(N_G(C')/C')$ except in the trivial case $p=2$ and $n=1$. Since the group $N_G(C')/C'$ has a normal cyclic p-Sylow subgroup of order p^n, it is known that $k(N_G(C')/C')$ decomposes into uniserial blocks of Loewy length p^n, and the number of simple modules in each block divides $p-1$. Therefore, the quiver of $k(N_G(C')/C')$ consists of disjoint cycles parametrized by the blocks of $k(N_G(C')/C')$, and each of them has as many vertices as the corresponding block has simple modules. Moreover, the relations are provided by the fact that all possible compositions of p^n arrows vanish. The Lemmata 2 and 4 now describe how these cycles have to be attached to the connected components of type $K_{n-1,e}$ which we already have by induction from $\bar{G}$. Thus the connected components of the Mackey quiver of G indeed are of type $K_{n,e}$. For this it finally remains to convince oneself that the relations involving the arrows α_0 are correct. Indeed, this is a straightforward consequence of the Mackey axiom using that all the arrows α_i are induced by the conjugation action of $a-1$. ∎

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